

An equivariant Kastler-Kalau-Walze type theorem

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Abstract

In this paper, we prove an equivariant Kastler-Kalau-Walze type theorem for spin manifolds without boundary. For 6 dimensional spin manifolds with boundary, we also give an equivariant Kastler-Kalau-Walze type theorem. Then we generalize this theorem to the general n dimensional manifold. An equivariant Kastler-Kalau-Walze type theorem with torsion is also proved.

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1 Introduction

The noncommutative residue found in [Gu] and [Wo] plays a prominent role in noncommutative geometry. Several years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which we call the Kastler-Kalau-Walze theorem. In [K], Kastler gave a brute-force proof of this theorem. In [KW], Kalau and Walze proved this theorem in the normal coordinates system simultaneously. In [A], Ackermann gave a note on a new proof of this theorem by means of the heat kernel expansion.

On the other hand, Fedosov et al defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace in [FGLS]. In [Wa1] and [Wa2], we generalized some results in [C1] and [U] to the case of manifolds with boundary. In [Wa3], We proved a Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator for 3,4-dimensional manifolds with boundary. Recently, Ponge defined lower dimensional volumes of Riemannian manifolds by the Wodzicki residue in [Po]. In [Da], an equivariant residue was defined. So it is nature question that whether we have an equivariant version of KKW theorem or not.

Atiyah-Singer index theorem is a very famous theorem. Its local version may be found in [BGV]. Considering a group action, Bismut defined the equivariant Bismut Laplacian instead of the Dirac Laplacian and proved the infinitesimal equivariant index theorem. Motivated by Bismut's proof, in this paper we also use the equivariant Bismut Laplacian instead of the Dirac Laplacian to give an equivariant Kastler-Kalau-Walze type theorem. Nextly, For 6 dimensional spin manifolds with boundary, we

also give an equivariant Kastler-Kalau-Walze type theorem. Then we generalize this theorem to the general n dimensional manifold. An equivariant Kastler-Kalau-Walze type theorem with torsion is also proved. In Section 2, we give an equivariant Kastler-Kalau-Walze type theorem. In Section 3, For 6 dimensional spin manifolds with boundary, we also give an equivariant Kastler-Kalau-Walze type theorem. In Section 4, we prove an equivariant Kastler-Kalau-Walze type theorem for a n dimensional spin manifold with boundary. In Section 5, an equivariant Kastler-Kalau-Walze type theorem with torsion is given.

2 An equivariant Kastler-Kalau-Walze type theorem.

Let M be a smooth compact Riemannian n -dimensional manifold without boundary and V be a vector bundle on M . Recall that a differential operator P is of Laplace type if it has locally the form

$$P = -(g^{ij}\partial_i\partial_j + A^i\partial_i + B), \quad (2.1)$$

where ∂_i is a natural local frame on TM and $g_{i,j} = g(\partial_i, \partial_j)$ and $(g^{ij})_{1 \leq i,j \leq m}$ is the inverse matrix associated to the metric matrix $(g_{i,j})_{1 \leq i,j \leq m}$ on M , and A^i and B are smooth sections of $\text{End}(V)$ on M (endomorphism). If P is a Laplace type operator of the form (2.1), then (see [Gi]) there is a unique connection ∇ on V and an unique endomorphism E such that

$$P = -[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L\partial_j}) + E], \quad (2.2)$$

where ∇^L denotes the Levi-civita connection on M . Moreover (with local frames of T^*M and V), $\nabla_{\partial_i} = \partial_i + \omega_i$ and E are related to g^{ij} , A^i and B through

$$\omega_i = \frac{1}{2}g_{ij}(A^j + g^{kl}\Gamma_{kl}^j\text{Id}), \quad (2.3)$$

$$E = B - g^{ij}(\partial_i(\omega_j) + \omega_i\omega_j - \omega_k\Gamma_{ij}^k), \quad (2.4)$$

where Γ_{ij}^k are the Christoffel coefficients of ∇^L .

Now we let M be a m -dimensional oriented spin manifold with Riemannian metric g . We recall that the Dirac operator D is locally given as follows in terms of orthonormal frames e_i , $1 \leq i \leq n$ and natural frames ∂_i of TM : one has

$$D = \sum_{i,j} g^{ij}c(\partial_i)\nabla_{\partial_j}^S = \sum_i c(e_i)\nabla_{e_i}^S, \quad (2.5)$$

where $c(e_i)$ denotes the Clifford action which satisfies the relation

$$c(e_i)c(e_j) + c(e_j)c(e_i) = -2\delta_i^j,$$

and

$$\nabla_{\partial_i}^S = \partial_i + \sigma_i, \quad \sigma_i = \frac{1}{4} \sum_{j,k} \langle \nabla_{\partial_i}^L e_j, e_k \rangle c(e_j)c(e_k). \quad (2.6)$$

Let

$$\partial^j = g^{ij} \partial_i, \quad \sigma^i = g^{ij} \sigma_j, \quad \Gamma^k = g^{ij} \Gamma_{ij}^k. \quad (2.7)$$

By (6a) in [Ka], we have

$$D^2 = -g^{ij} \partial_i \partial_j - 2\sigma^j \partial_j + \Gamma^k \partial_k - g^{ij} [\partial_i(\sigma_j) + \sigma_i \sigma_j - \Gamma_{ij}^k \sigma_k] + \frac{1}{4}r, \quad (2.8)$$

where r is the scalar curvature.

Let a compact group G act isometrically on M and preserve the spin structure. This action generates a Killing vector field X . Let L_X be the Lie derivation on the Spinors bundle. The Levi-Civita connection ∇^L lifts a Clifford connection ∇^S . Following Bismut, we define the equivariant Bismut Laplacian.

$$H_X = (D + \frac{1}{4}c(X))^2 + L_X; L_X = \nabla_X^S + \mu(X), \quad (2.9)$$

So

$$H_X = D^2 + \frac{1}{4}Dc(X) + \frac{1}{4}c(X)D + \nabla_X^S + \mu(X) - \frac{1}{16}|X|^2. \quad (2.10)$$

Let $X = X_j \partial_j$. So $L_X = X_j \partial_j + \frac{1}{4}X_j \sigma_j + \mu(X)$. Then by (10) in [Wa5], we have

$$\begin{aligned} H_X &= -g^{ij} \partial_i \partial_j + [X_j - 2\sigma^j + \Gamma^j + \frac{1}{4}c(\partial^j)c(X) + \frac{1}{4}c(X)c(\partial^j)]\partial_j \\ &+ g^{ij} [-\partial_i(\sigma_j) - \sigma_i \sigma_j + \Gamma_{ij}^k \sigma_k + \frac{1}{4}c(\partial_i)\partial_j(c(X)) + \frac{1}{4}c(\partial_i)\sigma_j c(X) \\ &+ \frac{1}{4}c(X)c(\partial_i)\sigma_j] + \frac{1}{4}r - \frac{1}{16}|X|^2 + \frac{1}{4}X_j \sigma_j + \mu(X). \end{aligned} \quad (2.11)$$

$$(D + \frac{1}{4}c(X))^2 = -[g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L \partial_j}^L) + E], \quad (2.12)$$

$$(D + \frac{1}{4}c(X))^2 + L_X = -[g^{ij}(\tilde{\nabla}_{\partial_i} \tilde{\nabla}_{\partial_j} - \tilde{\nabla}_{\nabla_{\partial_i}^L \partial_j}^L) + \tilde{E}], \quad (2.13)$$

By (2.2)-(2.4), we have

$$\tilde{\omega}_i = \omega_i + \frac{1}{2}g_{ij}X_j, \quad \tilde{A}_j = A_j + X_j, \quad \tilde{B} = B + \frac{1}{4}X_j \sigma_j + \mu(X). \quad (2.14)$$

By (11) in [wa5] and (2.2)-(2.4), we have

$$\tilde{E} = E + \frac{1}{4}X_j \sigma_j + \mu(X) - g^{ij} [\partial(\frac{1}{2}g_{jk}X_k) + \frac{1}{2}g_{il}X_l \omega_j + \frac{1}{4}g_{ik}g_{jl}X_k X_l - \frac{1}{2}g_{kl}X_l \Gamma_{ij}^k]. \quad (2.15)$$

$$E = -\frac{1}{4}r + \frac{1}{16}|X|^2 + \frac{1}{2}[e_j(\frac{1}{4}c(X))c(e_j) - c(e_j)e_j(\frac{1}{4}c(X))] + \sum_i \langle e_i, X \rangle^2. \quad (2.16)$$

By (2.14), we have

$$\tilde{\nabla}_Y = \nabla_Y + \frac{3}{4}g(X, Y). \quad (2.17)$$

By $\text{tr}\sigma_j = 0, \text{tr}\sigma^j = 0$ and direct computations, we get

$$\begin{aligned} \text{tr}\left(\frac{r}{6} + \tilde{E}\right) &= \dim(S(TM)) \left\{ -\frac{1}{12}r + \frac{1}{16}|X|^2 + \sum_i \langle e_i, X \rangle^2 \right. \\ &\quad \left. - g^{ij} \left[\partial_i \left(\frac{1}{2} g_{jk} X_k \right) + \frac{1}{2} g_{il} X_l \left[\frac{1}{2} g_{j\alpha} (\Gamma^\alpha + g^{kl} \Gamma_{kl}^\alpha) \right] \right] \right. \\ &\quad \left. - \frac{1}{4} \langle \partial^j, X \rangle g_{ij} + \frac{1}{4} g_{ik} g_{jl} X_k X_l - \frac{1}{2} g_{kl} X_l \Gamma_{ij}^k \right\} + \text{tr}(\mu(X)). \end{aligned} \quad (2.18)$$

Let $\dim M = m$. By [A], we know that

$$\text{Wres}(H_X^{-\frac{m}{2}+1}) = \frac{m-2}{(4\pi)^{\frac{m}{2}} \Gamma(\frac{m}{2})} \text{tr}\left(\frac{r}{6} + \tilde{E}\right), \quad (2.19)$$

where Wres denotes the noncommutative residue (see [A]). So by (2.18) and (2.19), we have

Theorem 1 *The following equality holds*

$$\begin{aligned} \text{Wres}(H_X^{-\frac{m}{2}+1}) &= \frac{m-2}{(4\pi)^{\frac{m}{2}} \Gamma(\frac{m}{2})} \int_M \left[\dim(S(TM)) \left\{ -\frac{1}{12}r + \frac{1}{16}|X|^2 + \sum_i \langle e_i, X \rangle^2 \right. \right. \\ &\quad \left. - g^{ij} \left[\partial_i \left(\frac{1}{2} g_{jk} X_k \right) + \frac{1}{2} g_{il} X_l \left[\frac{1}{2} g_{j\alpha} (\Gamma^\alpha + g^{kl} \Gamma_{kl}^\alpha) \right] \right] \right. \\ &\quad \left. \left. - \frac{1}{4} \langle \partial^j, X \rangle g_{ij} + \frac{1}{4} g_{ik} g_{jl} X_k X_l - \frac{1}{2} g_{kl} X_l \Gamma_{ij}^k \right\} + \text{tr}(\mu(X)) \right]. \end{aligned} \quad (2.20)$$

3 An equivariant KKW theorem for 6-dimensional manifolds with boundary

Let M be a 6-dimensional compact oriented spin manifold with boundary ∂M . We assume that the metric g^M on M has the following form near the boundary,

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \quad (3.1)$$

where $g^{\partial M}$ is the metric on ∂M . Let a compact group G act isometrically on M and preserve the spin structure. Near the boundary, the group action is not necessarily product action. Let $\widetilde{\text{Wres}}$ denote the noncommutative residue for manifolds with boundary (see [FGLS]). In the following, we want to compute $\widetilde{\text{Wres}}[\pi^+ H_X^{-1} \circ \pi^+ H_X^{-1}]$. By the definition of the noncommutative residue for manifolds with boundary, we have

$$\widetilde{\text{Wres}}[\pi^+ H_X^{-1} \circ \pi^+ H_X^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-6}(H_X^{-2})] \sigma(\xi) dx + \int_{\partial M} \Phi, \quad (3.1)$$

where

$$\begin{aligned} \Phi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \\ &\times \text{trace}_{S(TM)} [\partial_{x_n}^j \partial_{\xi'}^{\alpha} \partial_{\xi_n}^k \sigma_r^+(H_X^{-1})(x', 0, \xi', \xi_n) \times \partial_{x'}^{\alpha} \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(H_X^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (3.2)$$

where the sum is taken over $r - k - |\alpha| + l - j - 1 = -6$, $r \leq -2, l \leq -2$. Interior term comes from Theorem 1. We only compute the boundary term. By (2.11), we have the symbol $\sigma_{-2}(H_X)$ and $\sigma_{-3}(H_X)$ of H_X . By the symbol composition formula, we have

$$1 = \sum_{|\alpha|=0}^2 \frac{1}{\alpha!} \partial_{\xi}^{\alpha} (\sigma(H_X)) D_x^{\alpha} (\sigma(H_X^{-1})), \quad (3.3)$$

where $D_x^{\alpha} = (-i)^{|\alpha|} \partial_x^{\alpha}$. Let $\sigma(H_X) = p_2 + p_1 + p_0$, $\sigma(H_X^{-1}) = r_{-2} + r_{-3} + r_{-4} + \dots$, then we can get

$$r_{-2}(H_X^{-1}) = |\xi|^{-2}; p_1 r_{-2} + p_2 r_{-3} + \sum_j \partial_{\xi_j} D_{x_j} r_{-2} = 0. \quad (3.4)$$

By (2.11) and (3.4) and Lemma 1 in [Wa4], we get

$$r_{-3}(H_X^{-1}) = r_{-3}(D^{-2}) - \sqrt{-1} |\xi|^4 (X_j - \frac{1}{2} < X, \partial_j >) \xi_j, \quad (3.5)$$

where $r_{-3}(D^{-2})$ in Lemma 1 in [Wa4].

Now we can compute Φ , since the sum is taken over $-r-l+k+j+|\alpha| = -5$, $r, l \leq -2$, then we have the same five cases as in [Wa4]. By $r_{-2}(D^{-2}) = r_{-2}(H_X^{-1})$, we know that our cases 1 (i) (ii) (iii) equal cases (i) (ii) (iii) in [Wa4]. So the sum of these three cases is zero by [Wa4].

case II) $r = -2, l = -3, k = j = |\alpha| = 0$.

By (3.2) and an integration by parts and (19) in [Wa4], we get

$$\begin{aligned} \overline{\text{case II}} &= -\sqrt{-1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\pi_{\xi_n}^+ \sigma_{-2}(H_X^{-1}) \times \partial_{\xi_n} \sigma_{-3}(H_X^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\ &= \sqrt{-1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \{ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(H_X^{-1}) \times [\sigma_{-3}(D^{-2}) \\ &\quad - \sqrt{-1} |\xi|^4 (X_j - \frac{1}{2} < X, \partial_j >) \xi_j] \} (x_0) d\xi_n \sigma(\xi') dx' \\ &= \text{case II} + \sqrt{-1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \frac{i}{2(\xi_n - i)^2} (-\sqrt{-1}) |\xi|^4 (X_j - \frac{1}{2} < X, \partial_j >) \xi_j (x_0) d\xi_n \sigma(\xi') dx' \end{aligned} \quad (3.6)$$

where case II is in [Wa4]. By $\int_{|\xi'|=1} \xi_j \sigma(\xi') = 0$ for $j < m$ and the metric on M and the Cauchy integral formula, we get

$$\sqrt{-1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{i}{2(\xi_n - i)^2} (-\sqrt{-1}) |\xi|^4 (X_j - \frac{1}{2} < X, \partial_j >) \xi_j (x_0) d\xi_n \sigma(\xi') dx'$$

$$= -\frac{1}{32}\Omega_4 X_m \text{Vol}_{\partial M} \quad (3.7)$$

where Ω_4 is the canonical volume of the sphere S^4 .

case III) $r = -3, l = -2, k = j = |\alpha| = 0$

$$\overline{\text{case III}}) = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-3}(H_X^{-1}) \times \partial_{\xi_n} \sigma_{-2}(H_X^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.8)$$

Let $A = \sqrt{-1}|\xi|^4(X_j - \frac{1}{2} < X, \partial_j >) \xi_j$. So

$$\overline{\text{case III}}) = \text{case III}) + \sqrt{-1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ A \partial_{\xi_n} \sigma_{-2}(H_X^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.9)$$

Similarly to (23) in [Wa4], using the Cauchy integral formula, we get

$$\overline{\text{case III}}) = \text{case III}) + \frac{1}{32}\Omega_4 X_m \text{Vol}_{\partial M} \quad (3.10)$$

By [Wa4], we know that $\text{case II}) + \text{case III}) = 0$, so by (3.6) (3.7) and (3.10), we get $\Phi = 0$ So we get the following theorem

Theorem 2 *For 6 dimensional spin manifolds with boundary, the following equality holds*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ H_X^{-1} \circ \pi^+ H_X^{-1}] &= \frac{1}{4\pi^3} \int_M \left[\left\{ -\frac{1}{12}r + \frac{1}{16}|X|^2 + \sum_i < e_i, X >^2 \right. \right. \\ &\quad \left. \left. - g^{ij} \left[\partial_i \left(\frac{1}{2} g_{jk} X_k \right) + \frac{1}{2} g_{il} X_l \left[\frac{1}{2} g_{j\alpha} (\Gamma^\alpha + g^{kl} \Gamma_{kl}^\alpha) \right] \right] \right. \right. \\ &\quad \left. \left. - \frac{1}{4} \langle \partial^j, X \rangle g_{ij} + \frac{1}{4} g_{ik} g_{jl} X_k X_l - \frac{1}{2} g_{kl} X_l \Gamma_{ij}^k \right\} + \text{tr}(\mu(X)) \right]. \end{aligned} \quad (3.12)$$

Remark. We may extend this theorem to the higher dimensional case.

In order to get the nonzero boundary term, we use a function f on M to perturb $\widetilde{\text{Wres}}[\pi^+ H_X^{-1} \circ \pi^+ H_X^{-1}]$. We find only the term $\overline{\text{case I ii}}$ change and other terms does not change. We get

Theorem 3 *For 6 dimensional spin manifolds with boundary, the following equality holds*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ f H_X^{-1} \circ \pi^+ H_X^{-1}] &= \frac{1}{4\pi^3} \int_M f \left[\left\{ -\frac{1}{12}r + \frac{1}{16}|X|^2 + \sum_i < e_i, X >^2 \right. \right. \\ &\quad \left. \left. - g^{ij} \left[\partial_i \left(\frac{1}{2} g_{jk} X_k \right) + \frac{1}{2} g_{il} X_l \left[\frac{1}{2} g_{j\alpha} (\Gamma^\alpha + g^{kl} \Gamma_{kl}^\alpha) \right] \right] \right. \right. \end{aligned}$$

$$-\frac{1}{4}\langle\partial^j, X\rangle g_{ij} + \frac{1}{4}g_{ik}g_{jl}X_kX_l - \frac{1}{2}g_{kl}X_l\Gamma_{ij}^k\Big\} + \text{tr}(\mu(X))\Big] - \pi\Omega_4 \int_{\partial M} \partial_{x_n} f|_{x_n=0} d\text{Vol}_{\partial M}. \quad (3.13)$$

4 An equivariant general KKW theorem for manifolds with boundary

Let M be a $n = \bar{n} + 2$ -dimensional compact oriented spin manifold with boundary ∂M and \bar{n} is an even integer. We will compute $\widetilde{\text{Wres}}[\pi^+ H_X^{-1} \circ \pi^+ H_X^{-\frac{\bar{n}}{2}+1}]$. By the definition of $\widetilde{\text{Wres}}$ (see [FGLS]), we only compute the term

$$\begin{aligned} \Phi' &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \\ &\quad \times \text{trace}_{S(TM)}[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(H_X^{-1})(x', 0, \xi', \xi_n) \\ &\quad \times \partial_x^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(H_X^{-\frac{\bar{n}}{2}+1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (4.1)$$

where the sum is taken over $r - k - |\alpha| + l - j - 1 = -n$, $r \leq -2, l \leq 2 - \bar{n}$. Similar to Section 3 and [WW], we divide Φ' into five terms and the first three terms have the same expressions with the three term in [WW].

$\overline{\text{case}} II : r = -2, l = 1 - \bar{n}, k = j = |\alpha| = 0$

$$\overline{\text{case}} II) = -\sqrt{-1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(H_X^{-1}) \times \sigma_{1-\bar{n}}(H_X^{-\frac{\bar{n}}{2}+1})](x_0) d\xi_n \sigma(\xi') dx' \quad (4, 2)$$

Similar to (3.30) in [WW], we have

$$\sigma_{1-\bar{n}}(H_X^{-\frac{\bar{n}}{2}+1}) = \frac{\bar{n}-2}{2} \sigma_2^{-\frac{\bar{n}}{2}+2} \sigma_{-3}(H_X^{-1}) - \sqrt{-1} \sum_{k=0}^{\frac{\bar{n}}{2}-3} \partial_{\xi_\mu} \sigma^{-\frac{\bar{n}}{2}+k+2} \partial_{x_\mu} \sigma_2^{-1} (\sigma_2^{-1})^k. \quad (4.3)$$

By (3.30) in [WW] and (3.5), we have

$$\sigma_{1-\bar{n}}(H_X^{-\frac{\bar{n}}{2}+1}) = \sigma_{1-\bar{n}}(D^{-\bar{n}+2}) - \sqrt{-1} |\xi|^{-4} (X_j - \frac{1}{2} < X, \partial_j >) \xi_j \frac{\bar{n}-2}{2} \sigma^{-\frac{\bar{n}}{2}+2}, \quad (4.4)$$

So

$$\begin{aligned} \overline{\text{case}} II &= \text{case} II + \sqrt{-1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\frac{1}{2(\xi_n - i)^2} \right. \\ &\quad \left. \times |\xi|^{-4} (X_j - \frac{1}{2} < X, \partial_j >) \xi_j \frac{\bar{n}-2}{2} |\xi|^{-\bar{n}+4} \right] \sigma(\xi') dx' := \text{case} II + A, \end{aligned} \quad (4.5)$$

By $\int_{|\xi'|=1} \xi_j \sigma(\xi') = 0$ for $j < m$ and the metric on M and the Cauchy integral formula, we get

$$A = \frac{(2 - \bar{n})2^{\frac{\bar{n}}{2}-2}}{(\frac{\bar{n}}{2} + 1)!} X_n \Omega(S_{\bar{n}}) \left[\frac{\xi_n}{(\xi_n + i)^{\frac{\bar{n}}{2}}} \right]^{(\frac{\bar{n}}{2}+1)} \Big|_{\xi_n=i}. \quad (4.6)$$

$$\overline{\text{case III}}, r = -3, l = 2 - \bar{n}, k = j = |\alpha| = 0$$

$$\overline{\text{case III}} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-3}(H_X^{-1}) \times \partial_{\xi_n} \sigma_{2-\bar{n}}(H_X^{\frac{2-\bar{n}}{2}})](x_0) d\xi_n \sigma(\xi') dx'. \quad (4.7)$$

By (3.5) and (3.33) in [WW], we have

$$\begin{aligned} \overline{\text{case III}} &= \text{case III} - \sqrt{-1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ (-\sqrt{-1}|\xi|^{-4}(X_j - \frac{1}{2} < X, \partial_j >) \xi_j) \\ &\quad \times \partial_{\xi_n} \sigma_{2-\bar{n}}(H_X^{\frac{2-\bar{n}}{2}})](x_0) d\xi_n \sigma(\xi') dx' := \text{case III} + B \end{aligned} \quad (4.8)$$

Similar to case II, through some computations, we get

$$B = \frac{\frac{\bar{n}}{2} - 1}{(\frac{\bar{n}}{2} + 1)!} 2^{\frac{\bar{n}}{2}+1} X_n \Omega(S_{\bar{n}}) \left[\frac{\xi_n}{(\xi_n + i)^{\frac{\bar{n}}{2}}} \right]^{(\frac{\bar{n}}{2}+1)} \Big|_{\xi_n=i}. \quad (4.9)$$

By (4.5), (4.6), (4.8) and (4.9), we have

$$\Phi' = \Phi + A + B = \Phi + \frac{3n-6}{(\frac{\bar{n}}{2} + 1)!} 2^{\frac{\bar{n}}{2}-2} X_n \Omega(S_{\bar{n}}) \left[\frac{\xi_n}{(\xi_n + i)^{\frac{\bar{n}}{2}}} \right]^{(\frac{\bar{n}}{2}+1)} \Big|_{\xi_n=i}. \quad (4.10)$$

By Theorem 1 and (4.10) and (3.42) in [WW], we get

Theorem 4 *The following equality holds*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ H_X^{-1} \circ \pi^+ H_X^{-\frac{\bar{n}}{2}+1}] &= \frac{n-2}{(4\pi)^{\frac{n}{2}} \Gamma(\frac{n}{2})} 2^{\frac{n}{2}} \int_M \left[\left\{ -\frac{1}{12}r + \frac{1}{16}|X|^2 + \sum_i < e_i, X >^2 \right. \right. \\ &\quad \left. \left. - g^{ij} \left[\partial_i \left(\frac{1}{2} g_{jk} X_k \right) + \frac{1}{2} g_{il} X_l \left[\frac{1}{2} g_{j\alpha} (\Gamma^\alpha + g^{kl} \Gamma_{kl}^\alpha) \right] \right] \right. \right. \\ &\quad \left. \left. - \frac{1}{4} \langle \partial^j, X \rangle g_{ij} + \frac{1}{4} g_{ik} g_{jl} X_k X_l - \frac{1}{2} g_{kl} X_l \Gamma_{ij}^k \right\} + \text{tr}(\mu(X)) \right] \\ &\quad - \frac{\bar{n}-2}{\bar{n}+1} \frac{\pi i}{(\frac{\bar{n}}{2} + 2)!} 2^{\frac{\bar{n}}{2}-1} A_0 \Omega(S_{\bar{n}}) \int_{\partial M} K \text{dvol}_{\partial M} \\ &\quad + \frac{3n-6}{(\frac{\bar{n}}{2} + 1)!} 2^{\frac{\bar{n}}{2}-2} \Omega(S_{\bar{n}}) \left[\frac{\xi_n}{(\xi_n + i)^{\frac{\bar{n}}{2}}} \right]^{(\frac{\bar{n}}{2}+1)} \Big|_{\xi_n=i} \int_{\partial M} X_n \text{dvol}_{\partial M}, \end{aligned} \quad (4.11)$$

where K is an extrinsic curvature and A_0 is a constant (see (3.42) in [WW]).

5 An equivariant KKW theorem with torsion

Let T be a real three form on M . Let $D_T = D + T$ where T denotes the three form induces the Clifford action. Then D_T is self adjoint operator and

$$D_T^2 = \Delta + \frac{1}{4}r + \frac{3}{2}dT - \frac{3}{4}\|T\|^2, \quad (5.1)$$

Define the Bismut Laplacian with torsion $H_X^T = (D_T + \frac{1}{4}c(X))^2 + L_X$. So

$$H_X^T = H_X + \frac{3}{2}dT - \frac{3}{4}\|T\|^2 + \frac{1}{4}Tc(X) + \frac{1}{4}c(X)T, \quad (5.2)$$

so by the formulas (2.3) and (2.4), we get

$$E_{H_X^T} = E_{H_X} + \frac{3}{2}dT - \frac{3}{4}\|T\|^2 + \frac{1}{4}Tc(X) + \frac{1}{4}c(X)T, \quad (5.3)$$

We get

Theorem 5 *The following equality holds*

$$\begin{aligned} \text{Wres}((H_X^T)^{-\frac{m}{2}+1}) &= \frac{m-2}{(4\pi)^{\frac{m}{2}}\Gamma(\frac{m}{2})} 2^{\frac{m}{2}} \int_M \left[\left\{ -\frac{1}{12}r + \frac{1}{16}|X|^2 + \sum_i \langle e_i, X \rangle^2 \right. \right. \\ &\quad \left. \left. - g^{ij} \left[\partial_i \left(\frac{1}{2}g_{jk}X_k \right) + \frac{1}{2}g_{il}X_l \left[\frac{1}{2}g_{j\alpha}(\Gamma^\alpha + g^{kl}\Gamma_{kl}^\alpha) \right] \right] \right\} \right. \\ &\quad \left. - \frac{1}{4} \langle \partial^j, X \rangle g_{ij} + \frac{1}{4}g_{ik}g_{jl}X_kX_l - \frac{1}{2}g_{kl}X_l\Gamma_{ij}^k \right\} + \text{tr}(\mu(X) + \frac{3}{2}dT - \frac{3}{4}\|T\|^2 + \frac{1}{2}Tc(X)) \Big]. \end{aligned} \quad (5.4)$$

Bismut proved the local infinitesimal equivariant index formula by the Bismut Laplacian. Bismut also proved the local index theorem with torsion (see BGV, section 8.3 and [Bi]). By above the the Bismut Laplacian with torsion, we may prove a local infinitesimal equivariant index formula with torsion as following (details will appear elsewhere).

Theorem 6 *Let $dT = 0$ and $i_X T = 0$ and X is small, we have*

$$\lim_{t \rightarrow 0} \text{str}[\exp(-tH_X^T)(x, x)] d\text{vol}_M = (2\pi\sqrt{-1})^{-\frac{n}{2}} \widehat{A}(F_g^{-T}(X)), \quad (5.5)$$

where $F_g^{-T}(X) = R^{-T} + \mu(X)$ and $\mu(X)$ is the moment map.

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